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NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH
Section A

ELSEVIER

Feedback system for elimination of the transverse mode coupling instability

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Abstract

Transverse mode coupling (TMC) instability [1] is one of the major limitations of a single bunch current in large storage rings. This paper is devoted to a theoretical analysis of the effect of transverse feedback systems on TMC thresholds. It presents a space-time formalism for a hollow beam model and time-independent collective interaction. Then, in the framework of a simplified model, the principle of eliminating the TMC by means of a special transverse feedback system is outlined. This approach is then generalized in the form of an algorithm.

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1. Introduction

Starting from the space-time formulation of the TMC theory with averaged equations for the "hollow beam" model, the formalism is further developed for the practically important case of localized impedances.

An important reciprocal property of the eigenvalues for the TMC modes is proved for the general case. Finally, a general algorithm for the elimination of the TMC effect by a special arrangement of a transverse feedback system is developed.

First experiments with a simplified variant of the proposed feedback [8] proved the possibility of using it for an increasing TMCI threshold.

2. Averaged equations

Let us consider localized impedances and let us use the notation for the kick of the collective force:

$$\delta p = e^2(xW(\Delta s) + x'G(\Delta s)) , \tag{1}$$

where e is the electron charge, x and x' are the coordinate and angle of the leading forward particle, and W and G are functions of the distance between the forward and the test particle Δs.

This W-function notation differs by a factor $(-1/c)$ from the usual notation [2]; however, it is more convenient to use Eq. (1) for localized interaction of a bunch with elements of the vacuum chamber.

The second term gives the so-called "fast damping" with zero chromaticity. Its practical value is small in comparison with the collective tune shift, so we neglect it everywhere in this paper.

The mean dipole moment of the particles of the bunch can be written in the form $D = A\sqrt{\beta}\cos(\psi + \omega_b\tau)$, where A and ψ the "slow" amplitude and phase. Apart from the resonances and for small tune shifts (in comparison with the betatron tune) we can use equations averaged in time:

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$$\frac{\mathrm{d}A\,\mathrm{e}^{\mathrm{i}\psi}}{\mathrm{d}\tau}=\frac{\omega_{\mathrm{b}}\beta^{3/2}}{\mathrm{i}\gamma mc^{2}}\int_{0}^{T}F\,\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{b}}\tau}\frac{\mathrm{d}\tau}{T}=\overline{F}\,, \tag{2}$$

where the integration time T must be much larger than the betatron oscillation period.

In a localized force the dipole moment depends on the "fast" time; $D=A\sqrt{\beta}\cos(\omega_{\mathrm{b}}\tau+\psi)$; besides, the localized force itself depends on time (factor $c\delta(z-z_0)$). Averaging of the time-dependent expression $A\cos(\omega_{\mathrm{b}}\tau+\psi)$ using $\tau(\omega_{\mathrm{b}}\,\mathrm{d}\tau=\mathrm{d}z/\beta)$ yields:

$$\frac{1}{\mathrm{i}}\int_{0}^{T}A\cos(\omega_{\mathrm{b}}\tau+\psi)c\delta(z-z_0)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{b}}\tau}\frac{\mathrm{d}\tau}{T}=\frac{c}{2\mathrm{i}\beta\omega_{\mathrm{b}}T_0}A\exp(\mathrm{i}\psi)\,, \tag{3}$$

where T_0 is the revolution time. Hence $Dc\delta(z-z_0)$ in the force of the usual equations transforms into

$$(c/(2\mathrm{i}\sqrt{\beta}\omega_{\mathrm{b}}T_0))A\exp(\mathrm{i}\psi)\,.$$

Let us use for convenience the variable of the "slow" dipole moment $\mathcal{D}=A\,\mathrm{e}^{\mathrm{i}\psi}$. Then, in every point, there are two groups of particles for some synchrotron amplitude: one with negative and one with positive energy offsets (synchrotron phase $\pm\varphi_{\mathrm{s}}$). The corresponding betatron phase and amplitudes have index $+$ for the upper particles and $-$ for lower particles. The sum of the dipole moments $\mathcal{D}$ is

$$\mathcal{D}(\varphi)=A^{+}\,\mathrm{e}^{\mathrm{i}\psi^{+}}+A^{-}\,\mathrm{e}^{\mathrm{i}\psi^{-}}=\mathcal{D}^{+}+\mathcal{D}^{-}\,.$$

We can rewrite the total derivative in Eq. (2) through partial derivatives:

$$\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}=\frac{\partial\mathcal{D}}{\partial t}+\left(\frac{\partial\mathcal{D}^{+}}{\partial\varphi}-\frac{\partial\mathcal{D}^{-}}{\partial\varphi}\right)\omega_{\mathrm{s}}\,.$$

Here φ is the absolute value of the synchrotron phase; the minus sign arises from the different direction of motion, or, in other words, from different signs of the derivatives of the synchrotron phase absolute value $\varphi=|\varphi_{\mathrm{s}}|$ for particles with positive and negative energy offsets.

Now the total averaged equations are:

$$\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}=\frac{\partial\mathcal{D}}{\partial\tau}+\frac{\partial\overline{\mathcal{D}}}{\partial\varphi}\omega_{\mathrm{s}}=2\overline{F} \tag{4}$$

and

$$\frac{\mathrm{d}\overline{\mathcal{D}}}{\mathrm{d}\tau}=\frac{\partial\overline{\mathcal{D}}}{\partial\tau}+\frac{\partial\mathcal{D}}{\partial\varphi}\omega_{\mathrm{s}}=0\,, \tag{5}$$

where $\mathcal{D}$ and $\overline{\mathcal{D}}$ are the sum and the difference of the dipole moments for particles with sychrotron phase $\pm\varphi_{\mathrm{s}}$. $\overline{F}$ must be taken from Eq. (2). One can take $\overline{\mathcal{D}}$ from the second equation and substitute it in the first:

$$\omega_{\mathrm{s}}^{2}\frac{\partial^{2}\mathcal{D}}{\partial\varphi^{2}}-\frac{\partial^{2}\mathcal{D}}{\partial\tau^{2}}=-2\frac{\partial\overline{F}}{\partial\tau}\,. \tag{6}$$

For simplicity we treat only the case with zero chromaticity; however, it is easy to find the same equations with nonzero chromaticity. One can find details in Ref. [3].

For example, let us solve the simple model for a "hollow" beam (particles with the only synchrotron amplitude) and for constant wake ($W=$ const.). Let us find solutions which oscillate with frequency α. Let us substitute $\mathcal{D}(\varphi,t)=\mathrm{d}(\varphi)\,\mathrm{e}^{\mathrm{i}\alpha t}$ in the equations and, after cancellation of the exponent, we have ($W=$ const.):

$$d''+\frac{\alpha^{2}d}{\omega_{\mathrm{s}}^{2}}+\frac{\alpha Q}{\omega_{\mathrm{s}}^{2}}\int_{0}^{\varphi}\mathrm{d}(\varphi')\,\mathrm{d}\varphi'=0\,, \tag{7}$$

where $Q=Ne^{2}W\beta(z_0)/(2\pi\gamma mcT_0)$.

After normalization of α and Q on synchrotron frequency ω_{s},

$$d''' + \alpha^2 d' + \alpha Q d = 0\,. \tag{8}$$

For finding α we have to define boundary conditions.

For bunch edges $\overline{\mathcal{D}} = 0|_{\varphi=0,\pi}$ so $d' = 0|_{\varphi=0,\pi}$. The third condition is $d'' + \alpha^2 d = 0|_{\varphi=0}$, because of the zero force at the head of the bunch.

So we can find solutions for eigenfrequencies in the following way: for fixed Q and α the solution for $d(\varphi)$ is the sum of three exponents $d(\varphi) = C_1\,\mathrm{e}^{\lambda_1\varphi} + C_2\,\mathrm{e}^{\lambda_2\varphi} + C_3\,\mathrm{e}^{\lambda_3\varphi}$, where λ_i are the roots of the equation

$$\lambda^3 + \alpha^2\lambda + \alpha Q = 0\,.$$

Three boundary conditions yield for C_i:

$$\begin{aligned}
&C_1\lambda_1 + C_2\lambda_2 + C_3\lambda_3 = 0\,,\\
&C_1\lambda_1\,\mathrm{e}^{\lambda_1\pi} + C_2\lambda_2\,\mathrm{e}^{\lambda_2\pi} + C_3\lambda_3\,\mathrm{e}^{\lambda_3\pi} = 0\,,\\
&C_1(\lambda_1^2+\alpha^2) + C_2(\lambda_2^2+\alpha^2) + C_3(\lambda_3^2+\alpha^2) = 0\,,
\end{aligned} \tag{9}$$

where α is the eigenfrequency.

The determinant of these equations is a function of α, so after putting it to zero we get the required equation for d [6].

In Fig. 1 there are some solutions of this system $\det M = 0$ versus Q (the solutions were obtained with a numerical solver of transcendent equations).

One can see that, after merging of the eigenfrequencies of two modes, solutions with the same real tune and with positive and negative imaginary parts of the tune appear. For this case the threshold of instability is

$$Q_{\mathrm{th}} = \frac{N_{\mathrm{th}} e^2 W\beta(z_0)}{2\omega_s \pi\gamma m c T_0} \simeq 0.362$$

and

$$N_{\mathrm{th}} \simeq \frac{0.362\times 2\pi\omega_s\gamma m c T_0}{e^2 W\beta(z_0)}\,.$$

2.1. Averaged equations for modes

Let us take the averaged equations for the "hollow" beam (4, 5) for an arbitrary wake-function:

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}\tau} = \frac{\partial\mathcal{D}}{\partial\tau} + \frac{\partial\overline{\mathcal{D}}}{\partial\varphi}\,\omega_s = 2\overline{F}\,,\\
&\frac{\mathrm{d}\overline{\mathcal{D}}}{\mathrm{d}\tau} = \frac{\partial\overline{\mathcal{D}}}{\partial\tau} + \frac{\partial\mathcal{D}}{\partial\varphi}\,\omega_s = 0\,.
\end{aligned} \tag{10}$$

Here

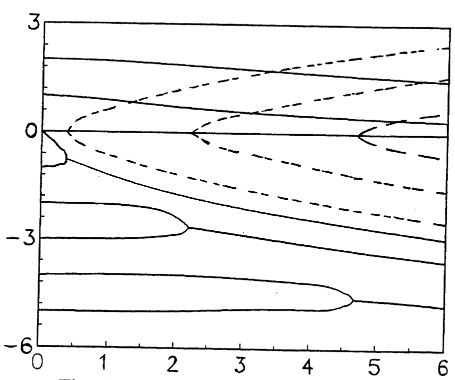

Fig. 1. Eigenfrequencies α versus Q.

$$2\bar{F} = -\frac{K}{\mathrm{i}}\int_0^{\varphi} W(a\cos\phi' - a\cos\phi)\mathscr{D}(\varphi')\,\mathrm{d}\varphi' ,$$

$$K = \frac{Ne^2\beta(z_0)}{2\pi\gamma mcT_0} .$$

For zero current for harmonics with number n the solution is:

$$\mathscr{D} = \mathrm{e}^{\mathrm{i}n\omega_s\tau}\frac{1}{2}d_n\cos n\varphi ,$$

$$\bar{\mathscr{D}} = -\mathrm{e}^{\mathrm{i}n\omega_s\tau}\frac{\mathrm{i}}{2}d_n\sin n\varphi ,$$

where d_n is an arbitrary constant.

Let us put in equations (10) the dipole moment in the form of an infinite sum of harmonics and with an oscillatory dependence on time:

$$\mathscr{D} = \mathrm{e}^{\mathrm{i}\alpha\tau}\frac{1}{2}\sum_{n=-\infty}^{\infty} d_n\cos n\varphi ,$$

$$\bar{\mathscr{D}} = \mathrm{e}^{\mathrm{i}\alpha\tau}\frac{\mathrm{i}}{2}\sum_{n=-\infty}^{\infty} d_n\sin n\varphi ;$$

then let us multiply first by $\cos n\varphi$, and second by $\sin n\varphi$ and let us integrate them over φ from zero to π. It yields for d_n:

$$\frac{\mathrm{i}\alpha\pi}{2}\frac{d_n + d_{-n}}{2} - \frac{\mathrm{i}n\omega_s\pi}{2}\frac{d_n - d_{-n}}{2} = \int_0^{\pi}\cos n\varphi 2\bar{F}\,\mathrm{d}\phi ,$$

$$\frac{\mathrm{i}\alpha\pi}{2}\frac{d_n - d_{-n}}{2} - \frac{\mathrm{i}n\omega_s\pi}{2}\frac{d_n + d_{-n}}{2} = 0 .$$

They hold for $n = 0$ too.

One can take the sum and the difference of these equations and express the force in terms of d_n:

$$(\alpha - n\omega_s)d_n = -K\frac{1}{\pi}\sum_{n'=-\infty}^{\infty} K_{nn'}d_{n'} , \tag{11}$$

where

$$K = \frac{Ne^2\beta(z_0)}{2\pi\gamma mcT_0} ,$$

$$K_{nn'} = \int_0^{\pi}\cos n\varphi\,\mathrm{d}\varphi\int_0^{\varphi} W(\Delta s)\cos n'\varphi'\,\mathrm{d}\varphi' .$$

Now let us obtain important properties for $K_{nn'}$:

$$K_{nn'} = (-1)^{n+n'}K_{n'n} .$$

2.2. Chess board symmetry of the mode coupling matrix

Let us begin with the hollow beam model, then integration in $K_{mm'}$ is done over a triangle domain: $0 \le \phi' \le \phi$, $0 \le \phi \le \pi$, according to the causal nature of the wake function (for $s > s'$, $W(s - s') = 0$):

$$K_{mm'} = \int_0^{\pi}\cos m\phi\,\mathrm{d}\phi\int_0^{\phi}\cos m'\phi' W(a\cos\phi - a\cos\phi')\,\mathrm{d}\phi' .$$

Interchanging the order of integration:

$$K_{mm'} = \int_0^{\pi} \cos m'\phi' \, d\phi' \int_{\phi'}^{\pi} \cos m\phi W(a \cos \phi - a \cos \phi') \, d\phi$$

and introducing new variables $\Phi = \pi - \phi$, $\Phi' = \pi - \phi'$:

$$K_{mm'} = \int_0^{\pi} \cos(m'\pi - m'\Phi') \, d\Phi' \int_0^{\Phi'} \cos(m\pi - m\Phi) W(a \cos(\pi - \Phi) - a \cos(\pi - \Phi')) \, d\Phi$$

$$= (-1)^{m+m'} k \int_0^{\pi} \cos m'\Phi' \, d\Phi' \int_0^{\Phi'} \cos m\Phi W(a \cos \Phi' - a \cos \Phi) \, d\Phi = (-1)^{m+m'} K_{m'm} ,$$

we come to a relation of "chess board symmetry" between the matrix elements:

$$K_{mm'} = (-1)^{m+m'} K_{m'm} .$$

Then we will use the equations for the "hollow" beam for a demonstration of the algorithm of elimination of the threshold of the strong head–tail effect.

For a nonhollow beam we have to divide the synchrotron plane into rings and use equations for an infinite number of rings and an infinite number of azimuthal modes for finding the eigen-frequencies. It was shown in Ref. [7], that the "chess board" property of coupling coefficients holds for coupling of rings with different synchrotron amplitudes, so it is general. The antisymmetry of the odd–even terms is the mathematical reason of merging of modes and TMC instability.

In the next section a more general approach is developed for exact (nonaveraged) equations.

3. General case

Let us start again with a hollow beam. The particles of a hollow beam occupy a ring in synchrotron phase space. Let us divide this ring into $2k + 1$ mesh elements with the nth one centred at the synchrotron phase ϕ_n:

$$\phi_n = \frac{2\pi n}{2k+1} - \frac{\pi}{2k+1} \quad (1 \le n \le 2k+1) .$$

Each mesh element comprises a group of real particles which are at this moment in this position of the phase space (and have a total dipole moment D_n and angle D'_n).

The division of the ring implies that all particles with the numbers n and $2k + 2 - n$ have equal synchrotron coordinates and opposite synchrotron "velocities" $\Delta E/E$, with the exception of the last particle in the bunch; this has no counterpart (see Fig. 2).

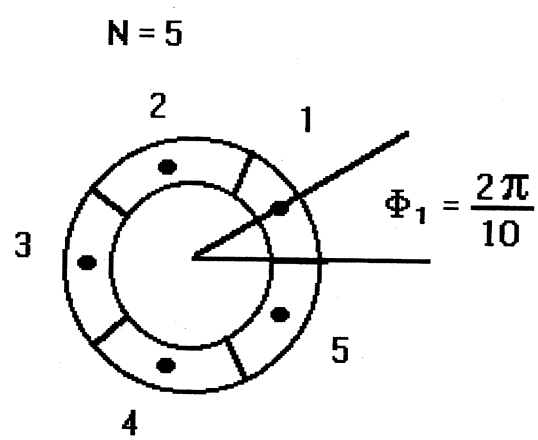

Fig. 2. Division of the longitudinal phase space into mesh elements for the hollow beam model.

Let us assume that the synchrotron tune is 2/5, for example. Then we have a 5×5 block matrix for 5 elements of the ring:

$$M=\begin{pmatrix} 0 & 0 & 0 & B & 0\\ 0 & 0 & 0 & 0 & B\\ B & 0 & 0 & 0 & 0\\ 0 & B & 0 & 0 & 0\\ 0 & 0 & B & 0 & 0 \end{pmatrix},$$

where

$$B=\begin{pmatrix} \cos\mu & \sin\mu\\ -\sin\mu & \cos\mu \end{pmatrix},$$

and μ is the betatron phase advance (we use normalized variables, so it has no β, etc.). The formula presented is evident: for the element of the mesh with number k for a 2/5 tune, the new dipole moment after a turn is just the transformed dipole moment of the $k-2$nd element.

The general matrix for arbitrary tunes is the so-called circulant matrix:

$$C_{ij}=\frac{\sin(2k+1)\varphi_{ij}}{(2k+1)\sin\varphi_{ij}}$$

with

$$\varphi_{ij}=\frac{1}{2}\left(\omega_s T-(2k+1-i+j)\frac{2\pi}{2k+1}\right).$$

It has the same (as in the previous case) form for tunes k/N, where N is the number of mesh elements. For other tunes it has small oscillations of different block's contributions in the dipole moments of the elements, but as for the eigenvectors and eigenfrequencies, they exactly correspond to first $-m\cdot\cdot m$ harmonics (the total number is $2m+1$). So it is the best solution for a finite number of divisions of the "hollow" beam to use the circulant matrix among all others, because we take into account exactly all possible lower modes.

3.1. Localized kick from collective force

For a localized kick we have the change of the transverse momentum of the trailing particle $\Delta p_\perp = \Sigma\, e\cdot DW(s-s')$, where the sum is taken over all forward particles of the bunch, instead of the same expression for the collective force for a distributed impedance.

If there are several localized impedances placed over a machine circumference, one should construct a kick matrix for each impedance and a transfer matrix for each azimuthal spacing so as to account for the mode phase advance in between the kicks. Then sequential multiplication of these matrices will result in a single-turn matrix which contains all the information on the stability of the $2k+1$ modes in question, neglecting their coupling with the truncated higher modes. From experience with the numerical computation, a good approximation for the eigenvalues is available with only few lower modes.

3.2. Two-dimensional mesh

For a nonhollow beam one can divide the synchrotron phase space into rings and transform the dipole moments and angles of all rings as done earlier for one ring of a "hollow" beam and with accurate account of the kick of the collective force. Eigenvalues of the obtained matrix are what we need for our problem. It was shown in Ref. [7], that it is possible to prove for the system obtained, that the general matrix for zero chromaticity is symplectic after some changing of variables, and reciprocal properties for the eigenvalues hold for the system obtained.

It was also shown there that it is possible to use this technique for a nonlinear potential well, etc. So now we have all mathematical tools for taking into account linear feedback of any kind, and for an explanation of the general principles of elimination of the TMCI threshold.

All following calculations are made with a code, based on the presented technique.

4. General principle of elimination of TMC instability

Feedback systems give the most universal cure for collective instabilities. Usually they treat multi-bunch and/or multi-turn instabilities where short bunches are considered as rigid and internal motion is disregarded. A feedback system, which cannot resolve the intra-bunch structure due to its limited bandwidth and acts equally on all particles in a short bunch, is apparently capable of curing the rigid bunch effects.

The TMC instability is of quite another nature; its essential feature is the ultra-relativistic ''causality'' of the interaction in the bunch: its leading particles are free from interaction with the collective field, while the trailing particles are affected by the intensity-dependent force. Compensation of such collective forces by means of a feedback system seems to be a complicated problem and is not feasible for very short bunches.

In what follows the feedback system will act equally on all the particles in the bunch, so that it couples only with the harmonic $m=0$, since all other harmonics in the basic set have zero dipole moment. Further we take into account feedback action in the following way: for the averaged equation a one turn feedback affects the real and the imaginary part in the zero harmonic tune, and no action on other harmonics. Due to coupling with zero harmonic they also interact with the feedback. It gives already some advantages of usual feedback. This was investigated in first works on TMCI with feedback [4,5]. We use oscillators (or oscillatory wake function feedback), which are also coupled with zero harmonic. It is evident how to take into account feedback action in real (localized on azimuth) form: in the pickup point we add the signal proportional to the dipole moment to the feedback; in the kicker point we transform the feedback signal from all the previous turns with the feedback wake-function and give a kick proportional to the obtained signal, equal for all particles.

Let us take two oscillators (modes) with antisymmetric linear coupling:

$$\begin{aligned} U' &= P_u \\ P_u' + \omega_u^2 U &= -kV , \\ V' &= P_v , \\ P_v' + \omega_v^2 V &= kU . \end{aligned} \qquad (12)$$

Here U, P_u and V, P_v stand for the canonically conjugate variables in any two coupled head–tail modes, and k is the relevant matrix element of the mode interaction matrix $K_{mm'}$, responsible for coupling these two modes (and proportional to the beam intensity).

The eigenfrequencies can be easily obtained:

$$\omega^2 = \frac{\omega_u^2 + \omega_v^2 \pm \sqrt{(\omega_u^2 - \omega_v^2)^2 - 4k^2}}{2} ,$$

and the threshold at $k = \frac{1}{2}(\omega_u^2 - \omega_v^2)$ is evident.

Disregarding the details of the transverse feedback system design and operation, we note that in principle its response function (we mean the transfer function between the pick-up signal and the kicker action) can have a rather general form. Here the feedback representation with an oscillator (or an LC-circuit) coupled to the $m=0$ mode, called U, will suffice. We take the feedback oscillator's coupling with the $m=0$ mode equal to k, but put it symmetrically into the set of equations, i.e., in a Hamiltonian manner. The feedback frequency is set equal to that of the mode V (say, with $m=-1$). According to what is stated above, the feedback oscillator Z is not coupled to the mode V. Then we have:

$$
\begin{aligned}
&U' = P_u\,,\\
&P_u' + \omega_u^2 U = -kV + kZ\,,\\
&V' = P_v\,,\\
&P_v' + \omega_v^2 V = kU\,,\\
&Z' = P_z\,,\\
&P_z' + \omega_v^2 Z = kU\,.
\end{aligned}
\tag{13}
$$

These equations correspond to the ones with small synchrotron and betatron phase advances over one revolution (equations from the previous section, but besides using a complex form for simplicity).

Simple calculations of the eigenvalues show that here the mode frequencies stay equal to their unperturbed values independently of k and hence do not depend on the beam current! Mode coupling is not totally eliminated: it is inherent in the current dependence of the eigenvectors. However, the mode frequencies can be made current-independent, hence they always remain real, which means elimination of the TMC instability.

For a multi-mode system (no matter how many modes coupled with the $m = 0$ mode there are) the principle shown in this example also works, with the exception of some degenerate cases.

The above consideration concerned the hollow beam. General cases have to be treated with much more complicated techniques and require a more general formalism, based on constructions of the previous section, to be presented in the following sections. Finally we will see that the idea of current-independent eigenfrequencies, achieved with a special transverse feedback system, can be extended to the general case.

5. General technique for elimination of current dependence of the TMC mode frequencies

In the previous section the possibility to obtain current-independent frequencies of the synchrobetatron modes by means of transverse feedback has been demonstrated on the basis of a simple model: only two modes and one feedback oscillator were considered with distributed interaction. Here we firstly present an algorithm for an arbitrary number of modes with the same interaction, then we will generalize the technique in order to account for localized collective force (and feedback) action.

5.1. Algorithm for averaged equations

Let us take averaged equations (11) and add to these equations for oscillators and one-turn feedback, which gives a real and an imaginary shift of the zero harmonic tune.

Let us use z_l for oscillator variables with frequencies $\bar{\omega}_l$ and coupling coefficients with zero harmonic q_l, $\bar{q}_l$.[1]

Let us show that N oscillators will suffice for changing frequencies of N higher modes and the zero mode, and make arbitrary values of all the $2N + 1$ frequencies. Our equation for zero harmonic is:

$$X_0' = \mathrm{i}\,\Delta X_0 - \mathrm{i}\sum_{i=0}^{N} K_{0i}X_i + \mathrm{i}\sum_{l=1}^{N} q_l z_l\,, \tag{14}$$

where Δ is the one-turn feedback coefficient.[2] Equations for oscillators are:

$$
\begin{aligned}
&z_1' - \mathrm{i}\bar{\omega}_1 z_1 = \mathrm{i}\bar{q}_1 X_0\,,\\
&\qquad\cdots\\
&z_N' - \mathrm{i}\bar{\omega}_N z_N = \mathrm{i}\bar{q}_N X_0\,.
\end{aligned}
\tag{15}
$$

[1] In the algorithm for eigenvalues calculation q_l, $\bar{q}_l$ are always multiplied, so they give only one coefficient.

[2] Prime means time derivative and summation begins from zero, because it does not matter here; this equation, after the substitution $X_0 \propto d_0\,\mathrm{e}^{\mathrm{i}\alpha}$, becomes the same as Eq. (11) for the zeroth mode, but here and below the coefficient K/π is included to K_{ij} for simplicity.

For finding eigenfrequencies ω put

$$X_i = A_i \, e^{i\omega t} ,$$
$$z_i = Z_i \, e^{i\omega t}$$

and we obtain for zero harmonic:

$$A_0(\omega - \Delta + K_{00}) = -\sum_{i=1}^{N} K_{0i} A_i + \sum_{l=1}^{N} q_l Z_l \tag{16}$$

and for oscillators:

$$\begin{aligned} Z_1(\omega - \bar{\omega}_1) &= \bar{q}_1 A_0 , \\ &\cdots \\ Z_N(\omega - \bar{\omega}_N) &= \bar{q}_N A_0 . \end{aligned} \tag{17}$$

From here we can express Z_l and put them in equations for the zero mode:

$$A_0\left(\omega - \Delta + K_{00} - \sum_{l=1}^{N} \frac{q_l \bar{q}_l}{\omega - \bar{\omega}_l}\right) = -\sum_{i=1}^{N} K_{0i} A_i . \tag{18}$$

New frequencies can be found from the following equation:

$$\det \tilde{B} = \begin{vmatrix} \Psi - \omega + K_{00} & K_{01} \cdots K_{0N} \\ K_{10} & \\ \cdots & M - \omega E \\ K_{N0} & \end{vmatrix} = 0 , \tag{19}$$

where

$$\Psi = -\Delta + \sum_{l=1}^{N} \frac{q_l \bar{q}_l}{\bar{\omega}_l - \omega} ,$$

and M is the coupling matrix of the zero harmonic with higher ones.

The form of Ψ with a common denominator is:

$$\Psi = -\Delta + \frac{F(\omega)}{\prod_{l=1}^{N} (\bar{\omega}_l - \omega)} ,$$

where $F(\omega)$ is a polynomial of Nth order.

The total determinant can be expressed through the first row (M_i are minors of elements K_{0i}):

$$(\Psi - \omega + K_{00})|M - \omega E| - \sum_{i=1}^{N} K_{0i} M_i(\omega) .$$

The collected form is:

$$H(\omega) \Big/ \prod_{l=1}^{N} (\bar{\omega}_l - \omega)$$

where H is a polynomial of $2N+1$st order.

In order to obtain $2N+1$ needed roots Ω_i, H must have the form:

$$H(\omega) = \prod_{i=1}^{2N+1} (\Omega_i - \omega)$$

and the corresponding equation is

$$(\Psi - \omega + K_{00})|M - \omega E| + \sum_{i=1}^{N} K_{0i} M_i(\omega) = \frac{H(\omega)}{\prod_{l=1}^{N} (\overline{\omega}_l - \omega)},$$

or

$$(\Psi - \omega + K_{00})|M - \omega E| = \frac{H(\omega)}{\prod_{l=1}^{N} (\overline{\omega}_l - \omega)} - \sum_{i=1}^{N} K_{0i} M_i(\omega),$$

and, finally

$$(\Psi - \omega + K_{00}) = \frac{H(\omega) - \left(\sum_{i=1}^{N} K_{0i} M_i(\omega)\right) \cdot \prod_{l}^{N} (\overline{\omega}_l - \omega)}{|M - \omega E| \prod_{l=1}^{N} (\overline{\omega}_l - \omega)}.$$

From the form of Ψ we see that the last equation holds if the nominator of the R.H.S. can be divided by an Nth order polynomial $|M - \omega E|$. Let x_k be all roots of this polynomial. So x_k are also roots of the nominator. Hence we have N equations for $N\overline{\omega}_l$.

From the R.H.S. of the previous equation we have a symmetrical polynomial of $\overline{\omega}_l$:

$$\prod_{l=1}^{N} (x_k - \overline{\omega}_l) = \frac{H(x_k)}{\sum_{i=1}^{N} K_{0i} M_i(x_k)},$$

where $k = 1, \ldots, N$; we can decompose it:

$$\prod_{l=1}^{N} (x_k - \overline{\omega}_l) = x_k^N - \sigma_1 x_k^{N-1} \cdots + (-1)^N \sigma_N .$$

Here $\sigma_1, \sigma_2, \ldots, \sigma_N$ are basic symmetrical functions of ω_l:

$$\begin{aligned} \sigma_1 &= \overline{\omega}_1 + \cdots + \overline{\omega}_N , \\ \sigma_2 &= \overline{\omega}_1 \overline{\omega}_2 + \cdots + \overline{\omega}_{N-1} \overline{\omega}_N , \\ &\cdots \\ \sigma_N &= \overline{\omega}_1 \times \cdots \times \overline{\omega}_N . \end{aligned}$$

When all N roots x_k are different, we obtain N linear equations for $\sigma_1, \sigma_2, \ldots, \sigma_N$. According to general algebraic laws $\overline{\omega}_1, \ldots, \overline{\omega}_N$ can be found as roots of an algebraic equation of Nth order:

$$z^N - \sigma_1 z^{N-1} \cdots + (-1)^N \sigma_N = 0 .$$

So, N unknown parameters among $2N + 1$ can be found separately.

Let us take the beginning form of Ψ:

$$\Psi = -\Delta - \sum_{l=1}^{N} \frac{q_l \overline{q}_l}{\overline{\omega}_l - \omega} .$$

Last $N + 1$ unknown coefficients $y_l = q_l \overline{q}_l$ and the one turn parameter Δ can be found from $N + 1$ linear equations, which appears when we substitute $N + 1$ needed ω in Eq. (19).

It is important to note that all the variables and parameters are complex, so all the frequencies may have damping.

5.1.1. Example

Let us demonstrate the algorithm for two modes. Here Δ is a one turn feedback parameter, $\bar{\omega}_1$ is an unknown oscillator frequency, and the coupling coefficient is $y_1 = q_1\bar{q}_1$. The characteristic equation reads:

$$\begin{vmatrix} -\Delta - \omega + K_{00} - \dfrac{y_1}{\omega - \bar{\omega}_1} & K_{01} \\ K_{10} & \omega_1 - \omega + K_{11} \end{vmatrix} = 0 . \tag{20}$$

For simplicity here $K_{00} = K_{11} = 0$ and $K_{10} = -K_{01} = q$; let us assign three values to the new frequencies $0, \omega_1, \omega_1$. We have:

$$\begin{vmatrix} -\Delta - \omega - \dfrac{y_1}{\omega - \bar{\omega}_1} & q \\ -q & \omega_1 - \omega \end{vmatrix} = \frac{-\omega(\omega_1 - \omega)^2}{\bar{\omega}_1 - \omega} , \tag{21}$$

or

$$\left(-\Delta - \omega - \frac{y_1}{\omega - \bar{\omega}_1}\right)(\omega_1 - \omega) + q^2 = \frac{-\omega(\omega_1 - \omega)^2}{\bar{\omega}_1 - \omega} . \tag{22}$$

According to the algorithm, the polynomial

$$(\omega_0 - \omega)(\omega_1 - \omega) - q^2(\bar{\omega}_1 - \omega)$$

must be divisible by $\omega_1 - \omega$; hence ω_1 is a root of the numerator and $\bar{\omega}_1 = \omega_1$. So for this example the oscillator frequency is equal to the frequency of the 1st harmonic.

After the substitution $\omega = \omega_0$ and $\omega = \omega_1$ in the first equation we have two equations for coefficients:

$$-\Delta(\omega_1 - \omega_0) - y_l + q^2 = 0 ,$$

$$-y_l + q^2 = 0 .$$

From them we obtain $\Delta = 0$ and $y_l = q^2$; this solution corresponds to the example which we used for the explanation of the main principle of eliminating the TMCI.

5.2. Algorithm for localized impedances and feedback

We can slightly transform the algorithm for a more realistic situation of localized impedances and feedback pickup and kicker.

Let us use the notation $A = \|a_{ij}\|$ for the matrix for $N+1$ modes; its order is $2(N+1)$ and the first two rows and columns correspond to a transformation of zero harmonic X_0. In Fig. 3 there are eigenmodes without feedback (the three modes model is used for calculations and $W =$ const).

Then $y_1, \ldots, y_N$ are coupling coefficients of zero harmonic and oscillators and Δ_1, Δ_2 are coefficients of one-turn feedback.

Let us use p, r, s, c as coefficients of free transformation of feedback oscillators. The total matrix of $2(2N+1)$ order for $N+1$ harmonics and N oscillators is

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & 0 & \cdots & & & \\ a_{21} & a_{22} & & & 0 & & & & \\ a_{31} & a_{32} & & & 0 & & & & \\ a_{41} & a_{42} & & & 0 & & & & \\ \cdots & & & \cdots & 0 & & & & \\ 0 & & & & p_1 & s_1 & 0 & 0 & \cdots \\ 0 & & & & r_1 & c_1 & 0 & 0 & \\ 0 & & & & 0 & 0 & p_2 & s_2 & \\ 0 & & & & 0 & 0 & r_2 & c_2 & \\ \cdots & & & & & & & & \cdots \end{pmatrix} , \tag{23}$$

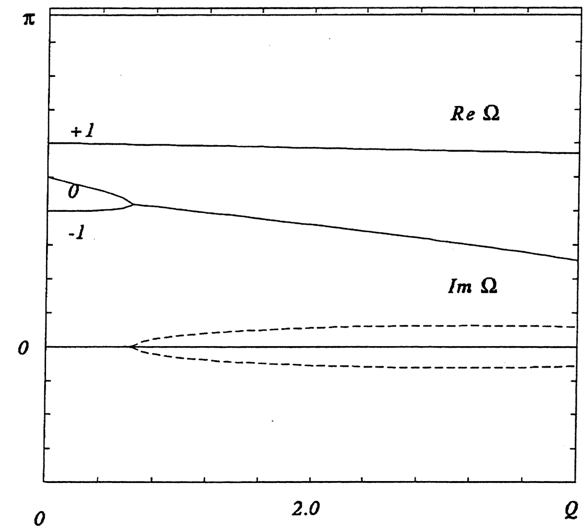

Fig. 3. Eigenfrequencies Ω versus $Q \propto I$. Parameter Q is normalized in such a way that for $Q = 1$ zeroth harmonic shift is equal to ω_s.

it must be multiplied by feedback matrix[3]

$$
\mathrm{FB} = \left(\begin{array}{cccccccccc}
\Delta_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \\
\Delta_2 & 1 & 0 & 0 & & 1 & 0 & 1 & 0 & \\
0 & 0 & 1 & 0 & & 0 & \cdots & & & \\
0 & 0 & 0 & 1 & & 0 & & & & \\
\cdots & & & & \cdots & 0 & & & & \\
0 & 0 & & & & 1 & 0 & 0 & 0 & \cdots \\
y_1 & 0 & & & & 0 & 1 & 0 & 0 & \\
0 & 0 & & & & 0 & 0 & 1 & 0 & \\
y_2 & 0 & & & & 0 & 0 & 0 & 1 & \\
\cdots & & & & & & & & & \cdots
\end{array}\right) \tag{24}
$$

and for $B = \bar{A} \cdot \mathrm{FB}$:

$$
\left(\begin{array}{ccccccccc}
\Delta_1 a_{11} + \Delta_2 a_{12} & a_{12} & \cdots & \cdots & a_{12} & 0 & a_{12} & 0 & \cdots \\
\Delta_1 a_{21} + \Delta_2 a_{22} & a_{22} & & & a_{22} & 0 & a_{22} & 0 & \\
\Delta_1 a_{31} + \Delta_2 a_{32} & a_{32} & & & a_{32} & \cdots & & & \\
\Delta_1 a_{41} + \Delta_2 a_{42} & a_{42} & & & a_{42} & & & & \\
\cdots & & & \cdots & & & & & \\
y_1 s_1 & 0 & \cdots & & p_1 & s_1 & 0 & 0 & \cdots \\
y_1 c_1 & 0 & & & r_1 & c_1 & 0 & 0 & \\
y_2 s_2 & 0 & & & 0 & 0 & p_2 & s_2 & \\
y_2 c_2 & 0 & & & 0 & 0 & r_2 & c_2 & \\
\cdots & & & & & & & & \cdots
\end{array}\right). \tag{25}
$$

Let us choose variables Δ_1, Δ_2, $y_i s_i$, $y_i c_i$, p_i, c_i. We can find them from $2(2N + 1)$ equations

[3] In this matrix the element 1, 1 corresponds to coordinate changing in point; it is the result of transforming feedback action from a point of the kicker to a point of the pickup.

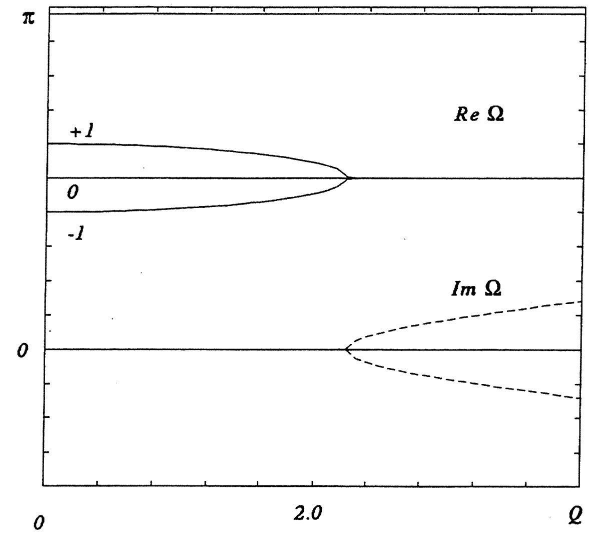

Fig. 4. Compensation of tune shift of mode with $m = 0$ by a one-turn feedback.

$$|B - \lambda_k I| = 0\,,$$

where λ_k are prescribed frequencies.

Coefficients Δ_1, Δ_2, $y_1 s_1$, $y_1 c_1$ are present linearly in the equations, and we can find them from the equations and substitute them in a previous set for simplification of the algorithm. Coefficients p_i, c_i can be found earlier from algebraic equations (equations for them are again symmetric).

One can see the results of applying the algorithm for a 3-mode model. Fig. 4 shows frequencies of 3 modes with optimal one turn feedback without oscillators. Fig. 5 corresponds to a total elimination of the threshold by adding two additional oscillators with frequencies of ± 1 modes.

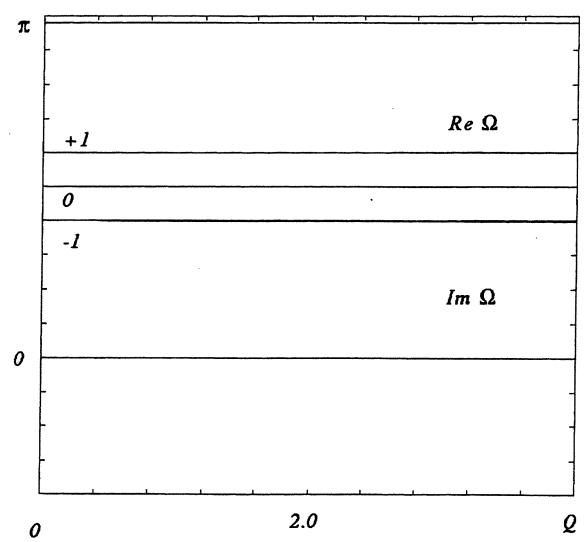

Fig. 5. Elimination of current dependence of mode frequencies.

5.3. Extension of the algorithm to radial modes

An important feature of the hollow beam model is that it contains only one unperturbed mode $m = 0$ with nonzero dipole moment, it has been essential for the evaluation of the determinant and eigenvalues in the above algorithm, which circumvents straightforward solving the high-order nonlinear set of equations.

Given a general distribution over the synchrotron amplitudes, an infinite number of unperturbed radial modes with nonzero dipole moment appears. According to the numerical procedure, the bunch is now divided into several rings, each specified by a certain value of the amplitude variable (i.e. the synchrotron amplitude of the ring) a. Hence, we obtain as many unperturbed modes with $m = 0$ as the amount of divisional rings we introduced.

However, we can always form a single linear combination of these modes to have a nonzero total dipole moment, and the complementary combinations, having zero total dipole moments. The substantiation comes from linear algebra: a linear space can be split into a direct sum of two subspaces:

i) a one-dimensional line $a\boldsymbol{v}$ with arbitrary a and $\boldsymbol{v} = (1, \ldots, 1)$;

ii) a hyperplane, spanned by all the vectors $\boldsymbol{x} = (\ldots, x_i, \ldots)$, which are orthogonal to $\boldsymbol{v}$, so that $(\boldsymbol{x} \cdot \boldsymbol{v}) = 0 \Rightarrow \Sigma\, x_i = 0$.

If x_i here denote the dipole moments of the rings, then all the combinations but one (namely, vector $\boldsymbol{v}$) have zero total dipole moment.

Such a change of the basic set (now consisting of the linear combinations, which themselves are not eigenvectors of the system) recovers the former situation, and the previous algorithm for the hollow beam is again valid, the role of the single mode carrying the dipole moment and, therefore, coupled to the feedback system, is taken over by the "total dipole" combination.

5.4. General linear feedback

In principle, we can construct an algorithm for arbitrary linear feedback. The behavior of the system "bunch + linear arbitrary feedback" is always determined by eigenvalues, but in the general case we have to use a matrix for transformation variables, consisting not only of variables in the present time, but also variables from the previous turns.

We only take a simple example of such a feedback. Let us imagine that we have N significant modes and that from previous turns we can calculate variables of these modes. We have the transformation matrix for N modes A; the total matrix can be obtained after multiplication by the kicker matrix:

$$K = \begin{pmatrix} \Delta_1 & \Delta_2 & \cdots & \Delta_{2N} \\ 0 & 1 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

which means changing of the angle of the zero harmonic proportionally to variables of the higher harmonics; Δ_i are unknown coefficients. For them we have $2N$ linear equations from conditions,

$$|KA - \lambda_i E| = 0\,,$$

where λ_i $(i \le 2N)$ are prescribed eigenfrequencies. So after solving the linear equations we obtain the unknown coefficients. Preceding calculations of modes variables can be made with a processor using data from previous turns, etc.

No doubt, that we can make the same algorithms for arbitrary linear feedbacks; the only problem arises from difficulties with finding coefficients and constructing complicated multiturn feedbacks.

6. Practical use of the algorithm

Here we give a few cases where the algorithm must be reconstructed.

Multiple roots can occur at the stage when we find the feedback oscillator frequencies from a high-order polynomial equation; these seriously complicate all the treatment. We can either avoid the multiplicity by tuning the assigned frequencies, or we must deal with the enlarged set of equations, including equations for zero derivative(s) of the polynomials at the multiple roots.[4]

Vanishing coupling of certain modes to the zeroth mode will apparently exclude them from a set of modes curable by feedback, because the only means to affect them with a feedback, which does not resolve intra-bunch motion, was to use the mode $m = 0$ as a mediator of the interaction. This problem has no solution at all.

An idealistic single-turn feedback whose action was implemented in the algorithm as a point-wise interaction with the single (thin lens) unknown parameter Δ, in addition to the $2N$ oscillator parameters. In large machines with a substantial collective phase advance between the single-turn feedback pickup and kicker, direct representation of a real feedback with such a model is no more possible. To compensate for the $m = 0$ mode tune slope we can use one more feedback oscillator. A first experiment with one oscillator instead of a one-turn feedback was made in LEP; see details in Ref. [8].

Damping of the feedback oscillators is needed to make the system more stable. However, for large damping previous turns are forgotten by the oscillator, so we always have to find the optimum, which depends on the machine parameters.

Noise is one of the restrictions of using multiturn feedbacks, because it leads to elimination of correlations of pickup signals with dipole moment after many turns.

7. Conclusion

In conclusion, we have to say that the algorithms are checked in simulations and partly (with one oscillator) in practice on LEP. The main result is that we can increase the TMCI threshold by means of a linear feedback of a special type, but algorithms must be chosen or upgraded according to real machine parameters.

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[4] We have in mind, that multiplicity of roots reduces the number of "useful" equations, however a corresponding number of equations is replenished from the condition, that the derivatives of the polynomial (up to a certain order) at such a point are zero, hence the number of equations will again suffice for the evaluation of the unknown feedback oscillators' frequencies and then their coupling coefficients.

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